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A Model of Sequential Location Among Foresighted Firms in Non-Uniform Density Markets

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Abstract

Startups vie for strategic positions in densely populated cities by paying high prices for rent, since the customer base they can establish is larger compared to suburban areas. For this paper, firms locate sequentially basing their decisions on correct expectations as to how their competitors locate and market-players face a non-uniform density function of customers. The solution is obtained using backward induction. Three types of market structures will be considered in this paper: duopoly, oligopoly, and perfect competition. The nature of these equilibriums differs from conventional papers in that firms face a uniform density of customers.

Introduction

The Hotelling model has been a cornerstone in the theory of location, which attempts to model or predict the location of firms throughout different industries. Hotelling (1929) studied the behavior of two firms, each one with a store in a one-dimensional market. In the model, he assumed that relocation would be cost-free and the firms would face a uniform distribution of customers. The non-cooperative outcome would find the duopolists, or the two sellers in the market involved in the sale of a specific product, located in the center of the market.

The purpose of this paper is to propose a change to one of the key underlying assumptions of the Hotelling model. The varying customer density functions across markets will be accounted for. There are many different types of customer density functions that firms in any market may face, including linear, concave, and convex functions. This paper is an attempt to use backward induction to find a solution to the basic cases. This method can be applied without loss of generality to any continuous, downward-sloping density functions.

Backward induction has been used as a tool to resolve cases whereby firms are assumed to face a uniform density function on the closed unit interval. The conclusion that can be derived is that firms spread themselves out over all possible locations. Despite this unrealistic assumption, the model is of great interest to many scholars in economics due to the appeal from its

simplicity. In this model, it is assumed that a sequential game of perfect information is played. In other words, only one firm moves at a time, and each firm knows every action of the other firms that moved at previous instances. In game theory terms, every information set contains only one node. For example, if firm three is supposed to make its moves after firm one and firm two have already moved, firm three will follow every move the previous firms have made and subsequently know the exact moment in which its turn has arrived. In this way, the game allows a single firm to choose its location at any one point in time. Furthermore, after each firm has decided on its location, it is assumed that the high expense of relocation will prohibit the firms from changing their minds and moving elsewhere. Therefore, each firm will decide on its optimal location based on two factors: a) the choices of previous firms, and b) the location rules that later firms will use. In the end, each firm takes into account the final configuration of the industry before making its choice. This paper attempts to explain how situations in which firms locate in different market structures are actually counterintuitive to what most economists would expect.

Three Entrants Facing a Non Uniform Density Market

Linear Density of Customers

The first example of a situation in which firms locate is the Hotelling spatial location problem with sequential entry (Prescott and Visscher, 1977) subject to linear density of customers. The firms have the option of locating at any point in the closed unit interval, while facing a density function of customers defined by $f(x) = -2Nx + 2N$ for $x \in [0, 1]$, where $N \in \mathbb{Z}^+$. In other words, the area under the curve represents the number of customers. In the model, we assume that buyers patronize the seller that is located closest to them and they each buy one unit of the good. Trade is conducted with a single homogeneous good and prices are set exogenously and priced equally at every shop.

Customers are also infinitely divisible. Following Hotelling's approach, entry will be limited to two firms. Following this will be study of the case of three firms present in the market. Sequentially, each firm chooses its optimal location in order to maximize profits, given the location of existing firms that have already chosen their locations. All firms have knowledge of the total number of firms allowed to participate in the market. Assuming that there are no fixed or variable costs for all firms, their final locations can be established using backward induction from the location decision of firm n to firm 1. When firm n selects its best location, it has to take into account the location of the $n - 1$ firms that have already located. Because each firm is a profit-maximizing agent, one particular rule must be adhered to when deciding where to locate. This can be called the profit-maximizing rule of location. In this symmetric information scenario, since firm $n - 1$ knows that the firm locating after it will want to maximize profits, firm $n - 1$ should use the location rule: firm n will predict its best location based on how the former varies its own position on the unoccupied spots. Because firm $n - 1$ can predict how firm n will locate based on its choices, the former can use the same rule to determine a profit-maximizing decision for itself based on the earlier occupied $n - 2$ positions. Firm $n - 2$ can use the same rule which it knows that firms n and $n - 1$ will apply when choosing their positions based on its location decision. Since firm $n - 2$ can predict how its competitors locating after it will choose their positions based on its location, it can formulate the location rule for itself as a function of the $n - 2$ already occupied positions. By executing this process, the location decision of the first firm can eventually be calculated. Therefore, the location of each firm that follows can be determined.

Next, it is important to consider the case in which entry is restricted to two or more firms, which is known specifically as a duopoly market structure. In this situation, firm two realizes that, after firm one has located, it then has the choice of locating on either side of firm one. The profit-maximizing rule of location in this instance is simply to choose the larger of the two disjoint intervals and position itself beside firm one. We maintain two of Hotelling's assumptions. First, no firm can locate on top of another firm. Second, all customers located between two firms are split equally among the firms. Because firm one knows that it will face an equal probability that firm two will locate an epsilon distance away on either side (where epsilon is infinitesimally small and positive), firm one will choose to locate at a (a is measured from zero) such that the number of customers it attracts to its right, which is given by $\int_a^1 -2Nx + 2Ndx$, is equal to the number of customers it attracts to it left, which is given by $\int_0^a -2Nx + 2Ndx$. Solving for a from the above equation, we find that firm one chooses to locate at $a = 0.293$.

Now, the focus can turn to the three-firm case or, more specifically, an oligopoly market structure. As shown by Prescott and Visscher (1977), a well-defined equilibrium exists assuming foresighted entry and the uniform density of customers. As expected, the outcome differs when we introduce the non-uniform density of customers. Because firms face a downward-sloping density curve, the problem is no longer symmetric. Let us begin by considering what happens when firm two locates to the right of firm one. We denote the locations of firm one, two, and three as a , b , and c , respectively; a , b , and c are measured from 0. Let us denote $x^+ \equiv x + \epsilon$, where x is any variable or real number, and ϵ is infinitesimally small and positive. Then, the optimal location of firm three conditional upon the existing locations of firm one and two is as follows:

1. If the payoff when firm three locates to the left of firm one, which is given by $\int_0^a -2Nx + 2Ndx$, is greater than the payoff when it locates to the right of firm two, which is given by $\int_b^1 -2Nx + 2Ndx$, and if the payoff to firm three is greater when firm three locates to the left of firm one as compared to locating to the right of firm one, which is given by $\int_a^{(b+a)/2} -2Nx + 2Ndx$, firm three will locate to the left of firm one. In short, if $\int_0^a -2Nx + 2Ndx > \int_b^1 -2Nx + 2Ndx$ and $\int_0^a -2Nx + 2Ndx > \int_a^{(b+a)/2} -2Nx + 2Ndx$, then firm three will locate to the left of a .
2. If the payoff to firm three when it locates to the right of firm one, which is given by

$\int_a^{(b+a)/2} -2Nx + 2Ndx$, is greater than its payoff when it locates to the left of firm one, which is given by $\int_0^a -2Nx + 2Ndx$, and if the payoff to firm three when it locates to the right of firm one is greater than its payoff when it locates to the right of firm two, which is given by $\int_b^1 -2Nx + 2Ndx$, firm three will locate to the right of firm one. In short, if $\int_a^{(b+a)/2} -2Nx + 2Ndx > \int_0^a -2Nx + 2Ndx$ and $\int_a^{(b+a)/2} -2Nx + 2Ndx > \int_b^1 -2Nx + 2Ndx$, then firm three will locate to the right of a .

3. If the payoff to firm three when it locates to the right of firm two, which is given by $\int_b^1 -2Nx + 2Ndx$, is greater than its payoff when it locates to the left of firm one, which is given by $\int_0^a -2Nx + 2Ndx$, and if the payoff to firm three when it locates to the right of firm two is greater than its payoff when it locates to the right of firm one, which is given by $\int_a^{(b+a)/2} -2Nx + 2Ndx$, firm three will locate to the right of firm two. In short, if $\int_b^1 -2Nx + 2Ndx > \int_0^a -2Nx + 2Ndx$ and $\int_b^1 -2Nx + 2Ndx > \int_a^{(b+a)/2} -2Nx + 2Ndx$, then firm three will locate to the right of b .
4. If locating to the left of a provides firm three with a smaller profit as compared to locating to the right of a and locating to the right of b , firm three will simply not choose the former.
5. Up to this point, we have already solved for firm three's optimal location as a function of firm one and firm two. Now, let us fix firm one's location by writing down the market share firm two is able to obtain conditional upon the location of firm one and its own location. Firm two, when deciding at which point to locate, would ask itself at which point on the unit interval it would be able to capture the largest market share given the location of firm one and how it expects firm three to locate. Given step four, an equilibrium solution will be obtained when firm two locates in a manner such that, when firm three enters, it is indifferent between the abovementioned two positions. Translating the aforesaid into mathematical equations, we obtain $\int_a^{(b+a)/2} -2Nx + 2Ndx = \int_0^a -2Nx + 2Ndx$ and $\int_b^1 -2Nx + 2Ndx = \int_0^a -2Nx + 2Ndx$. If we solve the simultaneous equations, the location of firm one and firm two are given by $a = (34 - 5\sqrt{34})/34$ and $b = (34 - 3\sqrt{34})/34$, respectively.
6. If $a > (34 - 5\sqrt{34})/34$, firm two will abandon the

previously mentioned location as firm one's location has changed. This is because firm two's profits are based on the location of firm one; since firm one's location changes, it must be that the location maximizes firm two's market share changes as well. If we repeat the abovementioned thought process again, we will see that firm two now chooses $b = ((34 - 3\sqrt{34})/34)^+$ as its optimal location.

7. Given these rules, the optimal location for firm one is $a = (34 - 5\sqrt{34})/34$ and its payoff is $\int_0^{(34-5\sqrt{34})/34} -2Nx + 2Ndx = 9N/34$. Given the location of firm one, firm two locates at $b = (34 - 3\sqrt{34})/34$ and its payoff is $\int_{(34-5\sqrt{34})/34}^1 -2Nx + 2Ndx = 16N/34$. Lastly, firm three positions itself to the right of firm one and its payoff is $\int_{(1-(2\sqrt{34})/17)}^{(34-5\sqrt{34})/34} -2Nx + 2Ndx = 9N/34$.

Now, let us investigate what happens when firm two locates to the left of firm one. By repeating the analogous thought process, the optimal locations of firms one and two are $a = (34 - 3\sqrt{34})/34$ and $b = (34 - 5\sqrt{34})/34$, and firm three locates to the right of firm two. The payoffs for firm one, two, and three are $16N/34$, $9N/34$, and $9N/34$, respectively.

In the second instance, firm one actually gets a larger payoff. Therefore, firm one would ensure that it locates at $(34 - 3\sqrt{34})/34$, and the two remaining firms will locate at the positions given by the solutions described above. By this fairly unsophisticated reasoning, it has been shown that there is a unique equilibrium whereby two firms will locate close to one another, and the last firm will locate far away from its rivals.

There is an additional conclusion that can be drawn from the above result. As expected, firm one obtains the largest slice of the market share. However, something unexpected happens if we examine the payoff to the other two firms. Although firm two has an advantage of locating earlier than firm three, it ends up with a payoff that is almost equal to that of firm three. To conclude, we see that a firm's ability to choose its location before its rivals does not necessarily imply a bigger payoff.

Non-linear density of customers

In the previous section, we were dealing with the assumption that firms face a linear distribution of customers. We now consider a different situation. The aim of this section is to consider the equilibrium positioning of the firms when they are faced with non-linear distributions of customers. The derivation follows from the

previous section, but all of the calculations are omitted. As we will discuss, the equilibrium outcome produces interesting results.

First, let us consider the case whereby three firms face a concave density function of customers defined by $f(x) = -x^2 + 0.5$ (See Fig. 1) for $x \in [0, \sqrt{0.5}]$. As a result, firm one locates at 0.205 and gets a payoff of 0.168. So, firm one will sell to 71.4 percent of all buyers. Following this, firm two locates at 0.068 and receives a payoff of 0.034. This translates into approximately 14.4 percent of the entire market. Finally, firm three positions itself at 0.034^+ and gets 0.033. Firm three captures the sales of 14.2 percent of all customers. Thus, we can conclude that firms two and three located next to one another with firm one locating further away.

Second, let us investigate what happens to the solution when the firms face a convex customer density function given by $g(x) = e^{-x}$ (See Fig. 2) for $x \in [0, 1]$. The optimal locations of firms one, two, and three are given by 0.627, 0.182, and 0.182^+ . Their corresponding payoffs are 47.4 percent, 26.3 percent and 26.3 percent of the entire market. Despite the fact that the qualitative location outcome is the same as observed in the concave density case, a major difference surfaces in the payoffs. Now, firm one gets a much smaller market share, while firms two and three enjoy a larger slice of the pie. Therefore, greater equity among firms is observed in this example.

The conclusion is that, in a sequence of entry, firms will locate first in the segment of the market where population density is relatively low. This guarantees that competing firms will not locate close to them, allowing them to capture all the sales in the area. Following this action by the first entrant, the remaining two firms will enter in areas whereby the density is higher so as to ensure that their profits are higher.

Infinite Entrants Facing a Linear Density Market

The method used in the above section becomes impossible to apply due to the amount of calculations involved when there are more than a few firms. However, despite the method's shortcomings, it can be modified to find a solution to the problem when there are an infinite number of potential entrants and each firm faces a fixed cost of entry. More specifically, this situation is a *perfectly competitive market structure*. Admittedly, it is unrealistic to assume the presence of an infinite number of potential entrants; then again, it is certainly interesting to observe what happens in this scenario. We denote \sim to be the market share of customers needed in

order for the firm to make zero profit. Therefore, $\lfloor 1/\alpha \rfloor$ is the maximum number of firms that this market will admit, where $\lfloor 1/\alpha \rfloor$ represents the greatest integer less than $1/\alpha$.

Due to the complexity of the problem, much of the literature has focused on trial and error to show that equilibrium exists when infinite entrants are allowed into the market. For example, in Prescott and Visscher (1977), a given market configuration was assumed to be an equilibrium. Following this, the authors proved that the suggested result was indeed an equilibrium. However, we will approach the problem in a completely different manner. We will only describe one of the many possible equilibria. The equilibria we will describe occur $\forall \alpha \in (0, 1)$ such that the last firm that enters the market will obtain a right-hand market share exactly equal to α . We will develop an algorithm that models how each subsequent firm that enters the market chooses its location given the location of other firms. Finally, we will combine the location behavior of all the firms in the market to obtain the final market configuration.

To evaluate whether a location is profitable or not, we will now have to redefine the profit maximizing rule of location since the situation has been altered. We know that firms stop entering if they cannot make at least α . So, at a certain point in time, no more firms will enter the market. We will now describe how the market would look under this scenario:

1. If two firms are located at Xa and Xb , where $Xa < Xb$, and no firm chooses to locate between them, this must imply that the payoff for any firm choosing to locate in between them will be $\int_{Xa}^{(Xa+Xb)/2} -2Nx + 2Ndx < \alpha$. This follows that no firm would want to locate in the closed interval $[Xa, Xb]$.
2. If a firm is located at x and no other firm chooses to locate to its left, then it must be that the payoff to any firm (if it chooses to locate to the left of the former) will be given by $\int_0^x -2Nx + 2Ndx < \alpha$, which ultimately implies that no firm would want to locate in the interval $[0, x]$. By similar reasoning, if the payoff for any firm that chooses to locate to the left of x is given by $\int_0^x -2Nx + 2Ndx > \alpha$, eventually there will be a firm that enters the market and will choose a point $y \in [0, x]$ such that $\int_0^y -2Nx + 2Ndx = \alpha$. But, this cannot be the case, as we assumed in the beginning that it was no longer profitable for any firm to enter the market.
3. If a firm is located at x and no firm chooses to locate to its right, then it must be that the payoff to any firm (if it chooses to locate to the right of the

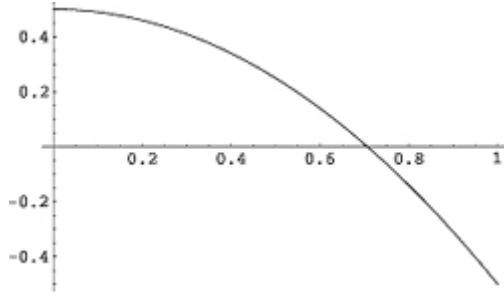


Figure 1: $f(x) = -x^2 + 0.5$

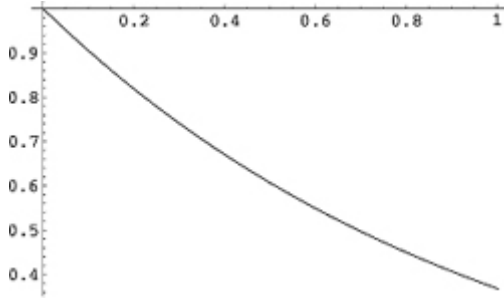


Figure 2: $g(x) = e^{-x}$

former) will be given by $\int_x^1 -2Nx + 2Ndx < \alpha$, which means that no firm would want to locate in the interval $[x, 1]$. But, if the payoff to any firm (if it chooses to locate to the right of the former) is given by $\int_x^1 -2Nx + 2Ndx \geq \alpha$, eventually a firm will choose to locate at $y \in [x, 1]$ such that $\int_y^1 -2Nx + 2Ndx = \alpha$. However, this cannot be the case, as we assumed in the beginning that it was no longer profitable for any firm to enter the market.

4. Each firm wants to position itself as closely as possible to the left end of the interval, considering the density of customers is highest there.

Before proceeding further, we will now define the set called profit (a collection of items or objects) a single firm can make in this market. We define X_i to be all feasible profits h that firm i can obtain, given that firm $1, 2, \dots, i-1$ have already located in this market. H is the total market share firm i can capture given its current location minus its costs, which are given by α . Given the above market configuration, a firm will be able to predict the upper bound of the total market share it can obtain and position itself to obtain this value conditional upon the set locations already occupied. We denote U_i to be the upper bound of profit firm i can obtain in the interval $[0, 1]$. It is trivial to see that $U_1 = \alpha$ for the first firm that en-

ters the market. Firm one wants to ensure that the market share it obtains to the left of it is exactly α . It stations at $X_1 = 1 - (\sqrt{N(\alpha - N)})/N$ and knows that if it chooses x_1 such that $\int_0^{x_1} -2Nx + 2Ndx > \alpha$, then eventually an incoming firm would position itself at $X_2 = X_1 - \epsilon$. This would occur because, when all other feasible locations enabling that entrant to make at least α are exhausted, the aforementioned location will at least enable the entrant to cover the cost of entry into the market. This implies that firm one would lose its entire left-hand market share. If firm one chooses x_1 such that $\int_0^{x_1} -2Nx + 2Ndx < \alpha$, then firm two would not want to position itself in the interval $[0, X_1]$. But in this instance, firm one is not making the maximum amount it can. Therefore, firm one will choose X_1 such that $\int_0^{X_1} -2Nx + 2Ndx = \alpha$.

In order to complete the explanation as to why $U_1 = \alpha$, we have to take the location of firm two into consideration. Assuming that firm one locates at the optimal position, firm two will position itself at X_2 to the right of firm one such that $\int_{X_1}^{(X_1+X_2)/2} -2Nx + 2Ndx = \alpha$. This is because firm two knows that if $\int_{X_1}^{(X_1+X_2)/2} -2Nx + 2Ndx > \alpha$, when there are subsequent entrants, they will position themselves at x_1^+ and firm two's market share will be reduced. And, if $\int_{X_1}^{(X_1+X_2)/2} -2Nx + 2Ndx < \alpha$, firm two is not positioning itself the best that it can in order to obtain the maximum left-hand market share. Since

we have argued that $\int_{X_1}^{(X_1+X_2)/2} -2Nx + 2Ndx$ cannot be less than α or more than α , by trichotomy $\int_{X_1}^{(X_1+X_2)/2} -2Nx + 2Ndx = \alpha$, firm two's optimal location will be revealed. Firm two's optimal location is given by $X_2 = (N - 2\sqrt{N(-2\alpha + N)} + \sqrt{N(-\alpha + N)})/N$. So, we have just proven that $U_1 = \alpha$, shown graphically in Figure 3. Please note that the shaded region in the graph is α .

Let us now calculate the upper bound of profits for firm two. We know that the left-hand market share of firm two is given by $\int_{(X_2+X_1)/2}^{X_2} -2Nx + 2Ndx$. By symmetry, firm two knows that firm three would use the abovementioned argument when deciding how far away to the right it should locate from firm two. So, it follows immediately that the right-hand market share of firm two is given by α , and $U_2 = \int_{(X_2+X_1)/2}^{X_2} -2Nx + 2Ndx$. In Figure 4, the locations of firm one and two based respectively on the upper bound of profits are displayed.

By repeating the entire thought process for subsequent firms that gain entry into the market, we can derive the upper bound of profit for firm n , which is given by $U_n = 3\alpha - 4(\sqrt{N(3\alpha + N(-1 + X_n)^2)} + N(-1 + X_n)(-1 + X_n))/9$ where $n > 1$. We can observe two phenomena when there are infinite entrants in the market. First, firms will be clustered very closely together at the left end of the closed interval where the density of customers is high and they are more spread out over the right end of the interval. Second, firms will enter the market sequentially from left to right.

Discussion

This paper has put forth explanations and illustrations of the optimal locations of firms in different market structures observed today in developing economies.

However, the analysis made here is simplified as compared to reality, given the assumption that the customer density function is simple and continuous. In the future, it would be useful to incorporate methods using density functions that model real demographic changes occurring in countries.

The results obtained here are based on the assumption that customer density functions faced by all firms are continuous. It would be intriguing to determine whether equilibriums exist for discontinuous customer density functions, and, if they do, the nature of these equilibriums. In addition, future study could attempt to examine the nature of the equilibriums given that infinite entry occurs and a finite number of firms are myopic in the way they choose to locate. In other words, their chosen location is not based on profit maximization but on random selection, while the remaining numbers of firms are going to maximize profit.

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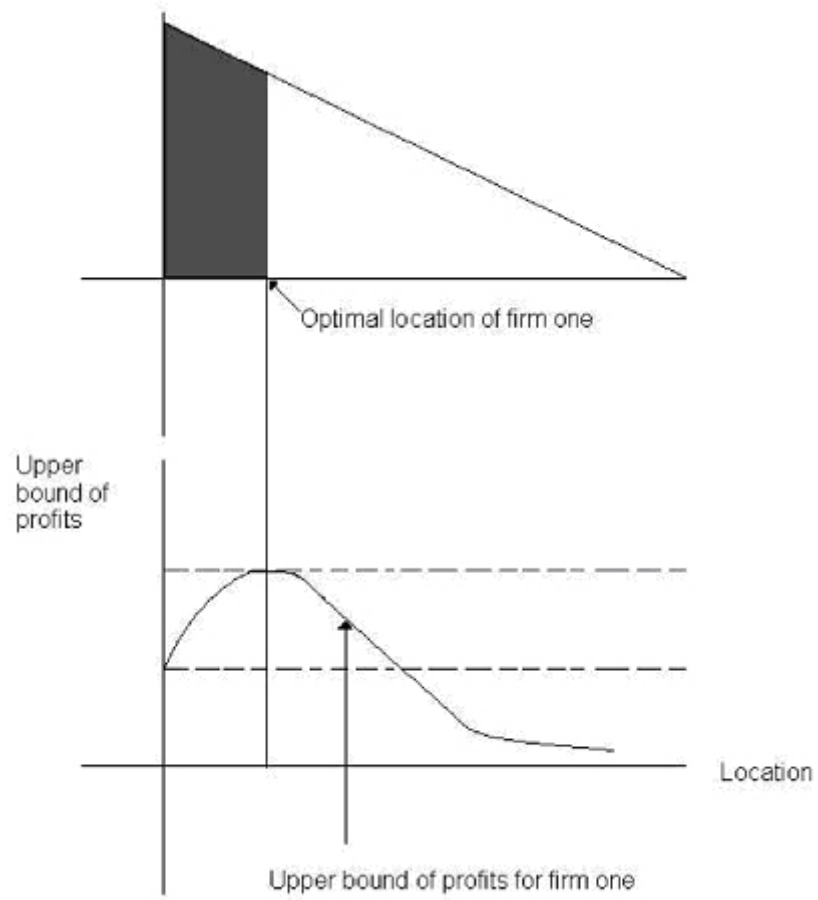


Figure 3: Optimal location of firm one based on profits

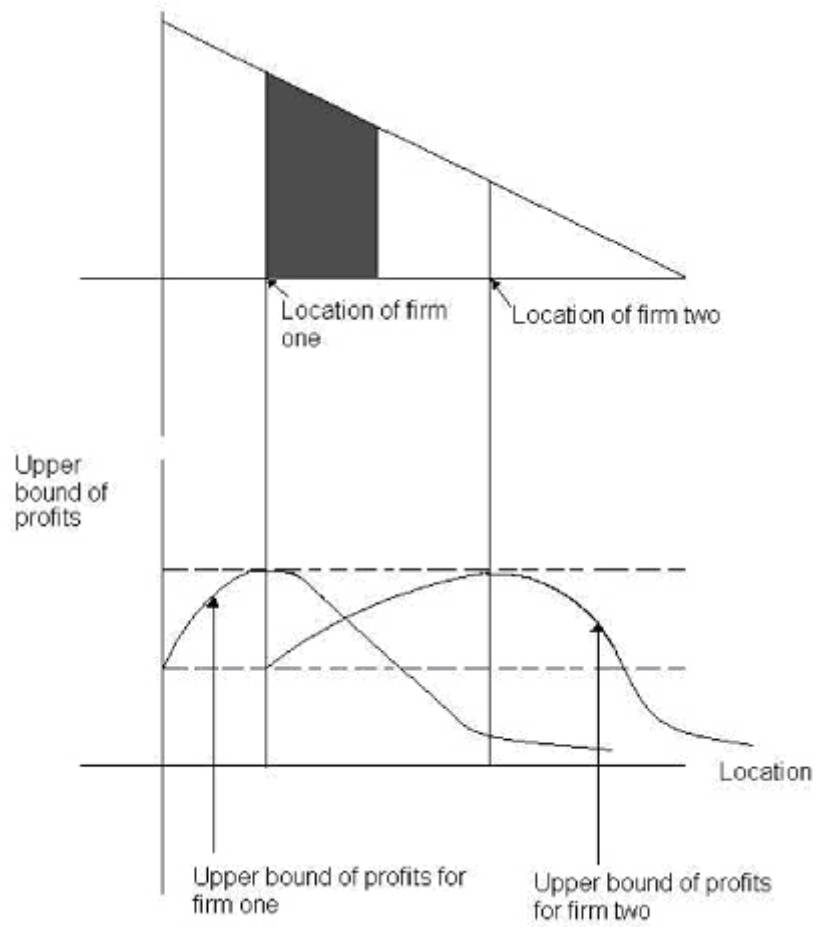


Figure 4: Optimal location of firm one and two based on profits