Shape Invariance in Supersymmetric Quantum Mechanics and its Application to Selected Special Functions of Modern Physics

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Abstract

We applied the methods of supersymmetric quantum mechanics to differential equations that generate well-known special functions of modern physics. This application provides new insight into these functions and generates recursion relations among them. Some of these recursion relations are apparently new (or forgotten), as they are not available in commonly used texts and handbooks. This method can be easily extended to explore other special functions of modern physics.

Introduction

The workhorse equation of non-relativistic quantum mechanics is the Schrödinger equation. Its time-independent form has a Hamiltonian operator H that yields eigenvalues E (allowed energies of a system such as an atom) and eigenfunctions (the wavefunctions from which to obtain other properties of the system, such as its probability of occupying a certain region of space or having a certain momentum). The Hamiltonian consists of a second derivative in the coordinate, representing the kinetic energy of the system, and a potential energy $V(x)$ representing the force(s) to which the system is subjected (such as the Coulomb force, or the harmonic oscillator force). The kinetic energy term is the same for all systems; it is the potential energy which distinguishes one problem from another. In one dimension, the Schroedinger equation is, in simplified units:

$$H\psi(x) = \left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x) \qquad (1)$$

Since V is a function of only the independent variable x, the Schrödinger equation is a second order linear differential equation with no first order term. In traditional modern physics and quantum mechanics courses, the equation is usually solved first by a brute force approach for finding the solutions to such differential equations. However, Dirac discovered a very elegant way to solve the problem using only operator algebra, more akin to Heisenbergs matrix formulation of quantum mechanics than to Schrödingers formulation. This approach was used to solve the harmonic oscillator problem (a good representation of many systems such as molecules), and is now taught alongside the differential equations approach. In the 1980s, an extension of this operator algebra approach was discovered and named supersymmetric quantum mechanics or SUSYQM (Witten 1981, Cooper and Freedman 1983, Cooper *et al.* 1995). It makes use of the particular form of the Schrödinger equation to construct operators, which allows for solution of many problems beyond just the harmonic oscillator, and even more surprisingly, the demonstration that apparently unrelated Hamiltonians have the same energy eigenvalues. As we shall demonstrate, SUSYQM can also yield insight into recursion relations of the functions, which appear in the solutions to the Schrödinger equation.

In solving the Schrödinger equation by the traditional brute force method, one encounters many different special functions with a myriad of properties, especially their recursion relations. Thus the hydrogen atom with its Coulomb force leads to wavefunctions involving Laguerre polynomials, while the harmonic oscillator leads to Hermite polynomials. In this paper, we provide another way of getting an insight into the properties of such functions. We employ the formalism of SUSYQM, which allows us to determine the spectrum of a quantum mechanical Hamiltonian and analyze it without actually solving the corresponding Schrödinger equation. Dutt *et al.* (Dutt *et al.* 1996) have studied the properties of the spherical harmonics using SUSYQM. We extend their work to two other sets of functions: the Laguerre and Hermite polynomials. Our analysis consists of converting the differential equations for these special functions into a Schrödinger type equation; namely, one with only a second derivative. Once we reach this familiar form, we then apply all the techniques of SUSYQM to analyze the solutions. In particular, we use several identities obtained from SUSYQM to generate recursion relations among Laguerre and Hermite polynomials. Some of these relations do not appear in textbooks or in the various handbooks which we have examined.

In Section II, we provide a brief introduction to SUSYQM. In Section III, we convert the differential equations of Laguerre and Hermite polynomials into Schrödinger equations by appropriate change of variable. Once in this form, these equations are amenable to the machinery of SUSYQM, which brings out the properties of, and relations among, their solutions. These relations are then translated back to yield recursion relations among the special functions with which we started. Finally, we discuss the pedagogic aspects of this work; in particular, its accessibility to undergraduates.

Supersymmetric Quantum Mechanics

For a quantum mechanical problem with a potential $V_-(x)$, supersymmetry allows one to construct a partner potential whose energy eigenvalues are in one-to-one correspondence with the excited states of $V_-(x)$; i.e., $E^{(+)}_{n-1} = E^{(-)}_n$ where $E^{(-)}_n$ are the eigenvalues of $V_-(x)$ and n is a positive integer. In SUSYQM, it is customary to describe $V_-(x)$ in terms of its ground state wavefunction, $\psi^{(-)}_0 (\equiv \psi_0)$, whose corresponding ground state energy, $E^{(-)}_0$ is adjusted to zero. The time-independent Schrödinger equation is then given by

$$H_- \psi_0(x) \equiv \left(-\frac{d^2}{dx^2} + V_-(x) \right) \psi(x) = 0 \qquad (2)$$

This gives $V_-(x) = \frac{\psi_0''}{\psi_0}$. Thus, the Hamiltonian can be rewritten in terms of the ground state wavefunction $\psi_0(x)$:

$$H_- = \left(-\frac{d^2}{dx^2} + \frac{\psi_0''}{\psi_0} \right)' \qquad (3)$$

We now define two operators:

$$A \equiv \left(\frac{d}{dx} - \frac{\psi_0'}{\psi_0} \right), \; A^\dagger \equiv \left(-\frac{d}{dx} - \frac{\psi_0'}{\psi_0} \right) \qquad (4)$$

The ratio $-\frac{\psi_0'}{\psi_0}$ is called the superpotential of the problem and is denoted by $W(x)$. Thus,

$$\psi_0(x) = exp\left(-\int_{x_0}^{x} W(x) dx \right) \qquad (5)$$

and $V_-(x) = W^2 - W'$. In terms of operators A and $A^\dagger$, the Hamiltonian $H_- \equiv A^\dagger A$. One can now define another operator $H_+ \equiv AA^\dagger = -\frac{d^2}{dx^2} + V_+(x)$, where $V_+(x) = W^2 + W'$. The potentials $V_\pm(x) \equiv W^2 \pm W'(x)$ are known as supersymmetric partners.

By construction, H_- and H_+ are Hermitian and positive semi-definite operators; i.e., their eigenvalues are either zero or positive. In fact, these Hamiltonians have the same eigenvalues (except for the ground state energy $E^{(-)}_0$, which is zero): $E^{(+)}_0 = E^{(-)}_1$, $E^{(+)}_1 = E^{(-)}_2$, ..., $E^{(+)}_{n-1} = E^{(-)}_n$ (Witten 1981). The ground state $\psi^{(-)}_0$ of $V_-(x)$ does not have a partner because $A\psi^{(-)}_0 = 0$.

Let us demonstrate this relationship between $E^{(-)}_n$ and $E^{(+)}_{n-1}$. We denote the eigenfunctions and eigenvalues of $H_\pm$ by $\psi^{(\pm)}_n$ and $E^{(\pm)}_n$ respectively. For $n \neq 0$,

$$\begin{aligned} H_+ \left(A\psi^{(-)}_n \right) &= AA^\dagger \left(A\psi^{(-)}_n \right) = A \left(A^\dagger A \psi^{(-)}_n \right) \\ &= AH_- \psi^{(-)}_n = E^{(-)}_n A\psi^{(-)}_n \end{aligned} \qquad (6)$$

and thus for positive integral values of n, $A\psi^{(-)}_n$ is an eigenfunction $\psi^{(+)}_n$ of H_+: $\psi^{(+)}_n \sim A\psi^{(-)}_n$, with eigenvalue

$$E^{(+)}_{n-1} = E^{(-)}_n \qquad (7)$$

where n is a zero or a positive integer and the constant of proportionality can be determined by normalizing both sides. Taking $\psi^{(+)}_{n-1} \equiv \alpha A \psi^{(-)}_n$,

$$\begin{aligned} 1 = \int \psi^{(+)*}_{n-1} \psi^{(+)}_{n-1} dx &= \alpha^2 \int \left(\psi^{(-)*}_n A^\dagger \right) \left(A\psi^{(-)}_n \right) dx \\ &= \alpha^2 \int \psi^{(-)*}_n A' A \psi^{(-)}_n dx \\ &= \alpha^2 E^{(-)}_n \int \psi^{(-)*}_n \psi^{(-)}_n dx = \alpha^2 E^{(-)}_n \end{aligned}$$

So $\alpha = \dfrac{1}{\sqrt{E^{(-)}_n}}$

Hence,

$$\psi_{n-1}^{(+)} = \frac{1}{\sqrt{E_n^{(-)}}} A\psi_n^{(-)} \tag{8}$$

Similarly

$$\psi_n^{(-)} = \frac{1}{\sqrt{E_{n-1}^{(+)}}} A^\dagger \psi_{n-1}^{(+)} \tag{9}$$

Eigenstates $\psi_n^{(-)}$ and $\psi_{n-1}^{(+)}$ are called supersymmetric partners. If the eigenvalues and the eigenfunctions of H_- are known, one can immediately solve for the eigenvalues and the eigenfunctions of Hamiltonian H_+. To summarize, SUSYQM generates two Hamiltonians and that have the same eigenvalues (except for the ground state of H_-) and related eigenfunctions. These relationships, however, do not guarantee the solvability of either potential. In principle, one would need to have found the solutions of one of the Hamiltonians by some standard method, in order to obtain the solution for the other. However, if these Hamiltonians have the additional property of shape invariance, one can determine all eigenvalues and eigenfunctions of both partners without solving their Schrödinger equations, as traditional methods requires.

Shape Invariance

If two partner potentials have a similar form, viz., they are similar functions of x and only differ in values of constant parameters and additive constants, they are said to be shape invariant (Infeld and Hull 1951,Gendenshtein and Krive 1985). The shape invariance condition is thus

$$V_+(x, a_0) = V_-(x, a_1) + R(a_0) \tag{10}$$

This can be generalized to

$$V_+(x, a_j) = V_-(x, a_{j+1}) + R(a_j) \tag{11}$$

where a_j and a_{j+1} are constant parameters and $A(a_j)$ is an additive constant. As an example, consider the superpotential $W(x) = -cot\ x$ with $b > 0$. The supersymmetric partner potentials generated by this superpotential are:

$$V_-(x, b) = W^2(x) - \frac{dW}{dx} = b(b-1)csc^2\ x - b^2$$

and

$$V_+(x, b) = W^2(x) - \frac{dW}{dx} = b(b+1)csc^2\ x + b^2$$

One can write

$$V_+(x, b) = V_-(x, b+1) + (b+1)^2 - b^2$$

Since the potentials $V_+(x, b)$ and $V_+(x, b+1)$ differ only by the value of the parameter (b vs. $b+1$) and the additive constant $(b+1)^2 - b^2$, they are shape invariant partners. As a specific example, choose $b = +1$. Then $V_-(x, 1) = -1$ and $V_+(x, 1) = 2csv^2\ x + 1$. $V_-(x, 1)$ is just the well-known infinite well potential taught in all undergraduate modern physics classes (Dutt *et al.* 1996), with bottom at -1. We thus deduce that the far less well known csc^2 potential, $V_+(x, 1)$ has the same energy eigenvalues despite appearing to be completely different. Most importantly, the shape invariance condition allows for the determination of the spectrum of each of the partner potentials algebraically without ever referring to their Schrödinger equations. Since the shape invariant potentials $V_+(x, a_j)$ and $V_-(x, a_{j+1})$ differ only by a constant, they share common eigenfunctions, and their eigenvalues are related by the same additive constants as the potentials themselves:

$$\psi_n^{(+)}(x, a_j) = \psi_n^{(-)}(x, a_{j+1}) \text{ and}$$
$$E_n^{(+)}(a_j) = E_n^{(-)}(a_{j+1}) + R(a_j) \tag{12}$$

Thus the eigenfunctions are relatev, from Eq. (9) by

$$\psi_{n+1}^{(-)} = \frac{1}{\sqrt{E_n^{(+)}(a_0)}} A^\dagger(x, a_0)\psi_n^{(+)}(x, a_0) \tag{13}$$

which by shape invariance (Eq. 12)

$$= \frac{1}{\sqrt{E_n^{(+)}(a_0)}} A^\dagger(x, a_0)\psi_n^{(-)}(x, a_2)$$
$$= \frac{1}{\sqrt{E_n^{(+)}(a_0)E_{n-1}^{(+)}(a_1)\dots E_0^{(+)}(a_n)}}\psi_0^{(-)}(x, a_{n+1}) \tag{14}$$

Thus, we have built a scheme for obtaining all the states $\psi_n^{(-)}$. Similarly for the eigenvalues, for example:

$$
\begin{aligned}
E_2^{(-)}(a_0) &= E_1^{(+)}(a_0) \\
&= E_1^{(-)}(a_1) + R(a_0) \text{ from Eq. (12)} \\
&= E_0^{(+)}(a_1) + R(a_0) \text{ from Eq. (7)} \\
&= (E_0^{(-)}(a_2) + R(a_1)) + R(a_0) \text{ again from Eq. (12)} \\
&= R(a_1) + R(a_0) \text{ since } E_0^{(-)}(a_i) = 0
\end{aligned}
$$

Therefore, the eigenvalues of $H_-(a_0)$ are generally given by

$$E_n^{(-)}(a_0) = \sum_{j=0}^{n-1} R(a_j) \tag{15}$$

Since from Eq. (5), $\psi_0^{(-)}(x, a_{n+1}) = exp\left(-\int_{x_0}^{x} W(x, a_{n+1})\right)$, one can determine all eigenstates of $H_-(x, a_0)$ using Eq. (14).

Study of Laguerre and Hermite Polynomials using SUSYQM

In this section, we will employ the machinery of SUSYQM to study the properties of Laguerre polynomials in detail and also briefly describe the results obtained for Hermite polynomials when studied similarly. The essential feature of our methodology is to rewrite the polynomials as products of two functions, to recast the defining differential equation (Laguerre or Hermite) in Schrödinger-like form; viz., eliminating the first derivative term. This then allows the application of the SUSYQM methods described in the previous section.

a. Laguerre polynomials

We start with the Laguerre differential equation:

$$x\frac{d^2 L_n^\alpha(x)}{dx^2} + (\alpha + 1 - x)\frac{dL_n^\alpha(x)}{dx} + nL_n^\alpha(x) = 0 \tag{16}$$

To cast this equation as a Schrödinger equation, we must eliminate the first derivative term. For that, we try the ansatz $L_n^\alpha(x) = f_n^\alpha(x)U_n^\alpha(x)$; each function will be determined later. Substituting this in the above differential equation, we get

$$
\begin{aligned}
&xfU^{''} + [2xf^{'} + (\alpha + 1 - x)f]U^{'} + \\
&[xf^{''} + (\alpha + 1 - x)f^{'} + nf]U = 0
\end{aligned} \tag{17}
$$

(We have suppressed the indices a and n for clarity.) At this point, we demand the vanishing of the first derivative term so that Eq. (17) resembles a Schrödinger equation; viz., an equation with a second derivative term in U and another term that looks like a product of a potential V and function U. This can be accomplished if the f-function satisfies the condition $2xf^{'} + (\alpha + 1 - x)f = 0$, whose solution is $f_n^\alpha(x) = c_n^\alpha x^{-\frac{(\alpha+1)}{2}} e^{\frac{\pi}{2}}$. Substituting this expression for $f(x)$ into Eq. 17, we get

$$-U^{''} + \left[\frac{(\alpha+1)(\alpha-1)}{4x^2} - \frac{(2n+\alpha+1)}{2x} + \frac{1}{4}\right]U = 0 \tag{18}$$

This equation has the form of the Schrödinger equation of a particle moving in an effective potential of a centrifugal term proportional to and a Coulomb potential proportional to . A SUSYQM analysis of this potential has been discussed in Witten 1981. In particular, Eq. (18) can be generated from the superpotential

$$W(x, n, \alpha) = \frac{2n + \alpha + 1}{2(\alpha + 1)} - \frac{(\alpha + 1)}{2x}$$

Corresponding partners V_- and V_+ are then given by:

$$
\begin{aligned}
V_-(x, n, \alpha) &= W^2 - W^{'} \\
&= \frac{(\alpha+1)(\alpha-1)}{4x^2} - \frac{(2n+\alpha+1)}{4x} \\
&+ \left[\frac{(2n+\alpha+1)}{2(\alpha+1)}\right]^2 \\
V_+(x, n, \alpha) &= W^2 + W^{'} \\
&= \frac{(\alpha+1)(\alpha+3)}{4x^2} - \frac{(2n+\alpha+1)}{4x} \\
&+ \left[\frac{(2n+\alpha+1)}{2(\alpha+1)}\right]^2
\end{aligned} \tag{19}
$$

They are related by the following shape invariance condition:

$$
\begin{aligned}
V_+(x, n, \alpha) &= V_+(x, n-1, \alpha+2) \\
&+ \left[\left(\frac{(2n+\alpha+1)}{2(\alpha+1)}\right)^2 - \left(\frac{(2n+\alpha+1)}{2(\alpha+3)}\right)^2\right]
\end{aligned} \tag{20}
$$

For the above superpotential W, the SUSYQM operators $A^\dagger$ and A are given by

$$A^\dagger = -\frac{d}{dx} + W = -\frac{d}{dx} + \frac{(2n+\alpha+1)}{2(\alpha+1)} + \frac{\alpha+1}{2x}$$

and

$$A = \frac{d}{dx} + W = \frac{d}{dx} + \frac{(2n+\alpha+1)}{2(\alpha+1)} + \frac{\alpha+1}{2x}$$

Thus, we started from the Laguerre differential equation and converted it into a Schrödinger equation. We found the corresponding potential described by a superpotential $W(x,n,\alpha) = \frac{2n+\alpha+1}{2(\alpha+1)} - \frac{(\alpha+1)}{2x}$ to be shape invariant. By SUSYQM, the solutions of this Schrödinger equation, $U_n^\alpha(x)$, are then related to each other via Eq. (14). We now explore the implications of this interrelationship to our original objects of study, the $L_n^\alpha(x)$ polynomials. We have found

$$L_n^\alpha(x) = f_n^\alpha(x) U_n^\alpha(x) = c_n^\alpha x^{-\frac{(\alpha+1)}{2}} e^{\frac{\pi}{2}} U_n^\alpha(x)$$

which yields:

$$U_n^\alpha(x) = K_n^\alpha x^{\frac{(\alpha+1)}{2}} e^{-\frac{\pi}{2}} U_n^\alpha(x) \qquad (21)$$

where $K_n^\alpha = \frac{1}{c_n^\alpha}$. To ensure that the $L_n^\alpha(x)$ have their traditional normalization (Merzbacher 1998), the normalization of $U_n^\alpha(x)$ leads to

$$K_n^\alpha = \sqrt{\frac{n!}{(2n+\alpha+1)[(n+\alpha)!]^3}}$$

Substituting the U from Eq. (21) into Eq. (14),

$$U_n^\alpha(x) = \frac{1}{\sqrt{E_{n,\alpha}^{(-)}}} A^\dagger(x,n,\alpha) U_{n-1}^{\alpha+2}$$

yields

$$K_n^\alpha x^{\frac{(\alpha+1)}{2}} e^{-\frac{\pi}{2}} L_n^\alpha(x) = -\frac{A_{n-1}^{\alpha+2}}{\sqrt{E_{n,\alpha}^{(-)}}} \Bigg[-\frac{d}{dx} + \frac{(2n+\alpha+1)}{2(\alpha+1)} - \frac{\alpha+1}{2x} \Bigg] x^{\frac{(a+3)}{2}} e^{-\frac{\pi}{2}} L_{n-2}^{\alpha+2}(x)$$

A tedious but straightforward simplification yields the desired result: the following recursion relation among Laguerre polynomials:

$$L_n^\alpha(x) = -\frac{(\alpha+1)}{n(n+\alpha+1)^2} \Bigg[\left(\frac{n+\alpha+1}{\alpha+1} \right) x - (\alpha+2) - x\frac{d}{dx} \Bigg] L_{n-1}^{\alpha+2}(x) \qquad (22)$$

An additional recursion relation is obtained via the identity

$$U_{n-1}^{\alpha+2}(x) = \frac{1}{\sqrt{E_{n,\alpha}^{(-)}}} A(x,n,\alpha) U_n^\alpha(x)$$

This gives

$$K_{n-2}^{\alpha+2} x^{\frac{(\alpha+3)}{2}} e^{-\frac{\pi}{2}} L_{n-2}^{\alpha+2}(x) = -\frac{K_n^\alpha}{\sqrt{E_{n,\alpha}^{(-)}}} \Bigg[\frac{d}{dx} + \frac{(2n+\alpha+1)}{2(\alpha+1)} - \frac{\alpha+1}{2x} \Bigg] x^{\frac{(a+1)}{2}} e^{-\frac{\pi}{2}} L_n^\alpha(x)$$

which upon simplification, yields

$$L_{n-2}^{\alpha+2}(x) = -(n+\alpha+1) \left[\frac{n}{x} + \frac{(\alpha+1)}{x} \frac{d}{dx} \right] L_n^\alpha(x) \qquad (23)$$

We have searched all commonly used mathematical tables and books on mathematical methods and did not find these identities. Equations (22) and (23) have thus been found by application of the methods of SUSYQM to the Laguerre equation, after transforming it into a Schrödinger-like equation.

b. Hermite polynomials

In this section, we will carry out a similar analysis for Hermite polynomials. We start with the Hermite differential equation

$$\frac{d^2 H_n(\xi)}{d\xi^2} - 2\xi \frac{dH_n(\xi)}{d\xi} + 2nH_n(\xi) = 0 \qquad (24)$$

We eliminate the first-order term by change of variable. Substituting the ansatz $H_n(\xi) = f_n(\xi)\psi_n(\xi)$ in the above differential equation, ($\psi_n(\xi)$ and $f_n(\xi)$ to be determined later), we get

$$\psi_n^{''} f_n + \psi_n^{'}[2f_n^{'} - 2f_n(\xi)] + \psi_n[f_n^{''} - 2f_n^{'}\xi + f_n(2n)] = 0 \qquad (25)$$

Setting the coefficient of term $\psi_n^{'}(\xi)$ equal to 0 gives $2f_n^{'} - 2f_n(\xi) = 0$, which is solved by $f_n(\xi) = c_n e^{\frac{\xi^2}{2}}$.

Replacing $f_n(\xi)$ in Eq. (9) and making the substitution $K \equiv 2n+1$ yields

$$-\psi_n^{''} + \xi^2\psi_n = K_n\psi_n \tag{26}$$

This equation is just the Schrödinger equation for the harmonic oscillator, and the corresponding superpotential is $W(\xi) = a\xi$, where a is a constant (Witten 1981). The SUSY partner potentials are given by $V_-(\xi,a) = a^2\xi^2 - a$ and $V_+(\xi,a) = a^2\xi^2 + a$. First, note that this set of potentials is shape invariant since

$$V_+(\xi,a) = V_-(\xi,a) + 2a$$

Inspection of the shape invariance condition reveals that $a_j = a$ for all j (This is the simplest possible example of SUSYQM: the case discovered by Dirac for the harmonic oscillator.) Secondly, an exact match with Eq. (26) requires that we set $a = 1$. Now the potential $V_-(\xi,a)$ differs from the traditional harmonic oscillator potential $V = \xi^2$ by an additive constant of -1. The associated energies of $V_-(\xi,a)$ and $V_+(\xi,a)$ can be derived by comparing their respective Schrödinger equations with Eq. (26):

$$E_{n-1}^{(+)} = E_n^{(-)} = E_n^{H.O.} - 1$$

Now let us obtain the recursion relations for the Hermite polynomials. The eigenfunctions $\psi_n \equiv \psi_n^{(\pm)}$ can be determined using the SUSYQM formalism. Since, $\psi_0^{(-)} = exp\left(-\int \xi d\xi\right) = exp\left(-\frac{\xi^2}{2}\right)$, one can find higher states using Eq. (14) and the supersymmetry operators A and $A^\dagger$:

$$A^\dagger(\xi,a_n) = \left[-\frac{d}{d\xi} + \xi\right] \text{ and } A(\xi,a_n) = \left[\frac{d}{d\xi} + \xi\right]$$

Since $H_n(\xi) = f_n(\xi)\psi_n(\xi) = c_n e^{\frac{\xi^2}{2}}\psi_n(\xi)$, inverting this relation gives

$$\psi_n(\xi) = A_n H_n(\xi) e^{-\frac{\xi^2}{2}} \tag{27}$$

where $A_n = \left(\frac{1}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}}$ in keeping with the traditional normalization of the Hermite polynomials (Merzbacher 1998). Substitution of the above SUSY operators $A(\xi,a_n)$, $A^\dagger(\xi,a_n)$ and the expression for the ground state into Eq. (14) leads to the following recursion relation:

$$H_n(\xi) = \left[2\xi - \frac{d}{d\xi}\right] H_{n-1}(\xi) \text{ and}$$
$$2nH_{n-1}(\xi) = \left[\frac{d}{d\xi}\right] H_n(\xi) \tag{28}$$

The identities of Eq. (28) are two of the familiar Hermite polynomial recursion relations (Merzbacher 1998), but obtained in a new way: using SUSYQM.

Conclusions

In this paper, we have demonstrated the application of supersymmetric quantum mechanics to the study of Laguerre and Hermite polynomials. We have derived apparently new recursion relations for the former, and found well-known ones for the latter by a new technique. This technique had earlier been applied to spherical harmonics, and can be applied to other sets of functions, provided that their differential equations can be recast as Schrödinger-like equations, whence potentials can be determined. Finally, we have observed that SUSYQM calculations such as these are excellent problems for undergraduate research.

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