JYI Journal of Young Investigators

The following information can be used when citing this article:

Bibliographic details

- Article name: Mathematical Modeling of Exponential Growth with Impulsive Perturbations and Applications in Ecology
- Author: Kraeva, Antoaneta
- Publisher: The Journal of Young Investigators
- Volume: 12
- Issue: 3
- Date published: 28 February 2005
- Link: `http://www.jyi.org/research/re.php?id=106`

Mathematical Modeling of Exponential Growth with Impulsive Perturbations and Applications in Ecology

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28$^{\text{th}}$ February 2005

Abstract

This paper studies the dynamics of any population growing in a bioreactor with instantaneous perturbations caused by taking away or adding a biomass at certain moments. The case when the perturbations are of negligible length is investigated, and impulsive differential equations are used for modeling the situation. The studied populations either grow or decay exponentially between two consecutive perturbations. Several different models depending on the type of perturbations are examined; for each model, existence results and explicit formulas for the solutions are obtained. Theoretical results are applied to several real-life cases to predict the population sizes in time, as well as the instant and the amount of the perturbations. We can find a means to control the considered ecosystem using the obtained results for the new models, and save time and money for conducting experiments and measuring population size.

Introduction

Modeling of population dynamics is an essential part of both research of and management of ecological systems (Murray 2002). A model is a mathematical description of the dynamics of a real process or a phenomenon. A model may be as simple as an equation with just one variable or as complex as a computer program with thousands of lines of code. One of the most difficult decisions when building a model is determining the model's appropriate level of complexity with regards to the given situation. Simple models are easier to understand and more likely to give insights that are applicable in a wide range of situations. However, they also have more simplistic assumptions and lack realism when applied to specific cases. Thus, they cannot be used to make reliable forecasts in practical situations. Including more details makes a model more realistic and easier to apply to specific cases. Unfortunately, in most practical cases available data is limited and permits only the simplest models.

At the same time, models are very useful not only for theoretical investigations of the dynamics of the system, but also for controlling the system in the best way at the right time to achieve the most beneficial impact.

In ecology, there are many known mathematical models describing different types of populations and behaviour (Ginberg *et al.* 1986; Turchin 2001). In the case of overlapping generations, differential equations with continuous solutions are usually used.

In the real world, we rarely see continuous behavior of the population size for a long time. Because population sizes are usually curbed by diseases, predators, or other environmental factors, some individuals from the population are "taken away" or "added" resulting in changes in the growth pattern. Different types of perturbations or disturbances can affect the population. In some cases, the population is subject to short-term perturbations whose duration is negligible in comparison to the lifetime of the population. Therefore, we can assume that these perturbations act instantaneously or in the form of impulses.

It is known that many biological phenomena involving thresholds, such as bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics, and frequency modulated systems, exhibit impulsive effects. In these cases, the continuous models are not adequate for describing the dynamics of the population, and it is necessary to apply a different type of differential equation - Impulsive Differential Equations. A theoretical description and a study of Impulsive Differential Equations are given by Lakshmikantham *et al.* (1989). Applying these equations to modeling a system gives us new models with piecewise continuous functions as solutions. The properties of the impulsive and the classical continuous models differ greatly, so deep investigations of the impulsive models are required.

This paper focuses on the modeling of populations under the influence of short-term perturbations in the case when populations are exponentially growing or decaying between consecutive instantaneous changes. Several different impulsive models with different types and instants of perturbations are studied. In each case, theoretical results such as existence, periodicity, boundedness, and explicit formulas for the solutions are proven. The results are then applied to a population of *Wolffia microscopica*, commonly known as duckweed. Already knowing the reproduction rate of this species of plants, we can estimate the population size at any given moment, and the best instants and amounts of perturbations for each model. We can also find the moments and quantities of changes needed to keep the process periodic in time. In this way we can easily control the population size by reducing the time and resources required to conduct experiments and study the chosen organisms.

Continous Exponential Growth or Decline

Definition 1: A population is any group of organisms coexisting at the same time and place that are capable of interbreeding.
Definition 2: The size of the population is the number of individuals.

The size is the basic measurable characteristic of a population. Usually it varies in time t and is denoted by N(t).

We will deal with the dynamics of species that reproduce and die over long time intervals and will look at long term population change in order to find an overall pattern. We will assume that the birthrate and the deathrate remain constant over time and that there is no environmental limitation. In the case of overlapping generations, the dynamics of the population could be described with an exponential model. This model is one of the basic principles of ecology, and is often referred to as the Malthusian law (Ginberg 1986, Turchin 2001). The dynamics of a population size N(t) are given by the initial value problem for the differential equation:

$$\frac{dN}{dt} = rN, N(0) = N_0 \tag{1}$$

where $N(t) > 0$ is the population size at the moment t, r =constant is the net rate of growth (births + immigration - deaths - emigration) per individual in the population per unit of time, and $N_0 > 0$ is the initial size of the population.

The Malthusian population model predicts either population growth without bound or inevitable extinction. The difference is based on whether the growth rate r (births + immigration - deaths - emigration) is positive or negative.

Case 1. Let $r > 0$ (i.e., the total amount of births and immigration is more than the total amount of deaths and emigration).

From the properties of the exponential function it follows that the function N(t) is increasing and that, therefore, the population grows in time. The graph of the exponential growth with r=0.1 and N(0)=1 is given in Figure 1.

Case 2. Let $r < 0$ (i.e., the total amount of births and immigration is less than the total amount of deaths and emigration).

Then the function N(t) is decreasing and the population decays in time. The graph of the exponential decay with r = -0.1 and N(0)=100 is given in Figure 2.

Case 3. Let $r = 0$ (i.e., the total amount of births and immigration is equal to the total amount of deaths and emigration).

The function $N(t) = N_0$ is a constant and the population is not changing in time.

The Malthusian model has many applications besides population growth. For example:

Biology: Bacteria in a culture dish will grow exponentially until the available food is exhausted. A new virus (SARS, West Nile, smallpox) will spread exponentially. Each infected person can infect multiple new people.

An atomic reaction: Each uranium atom that undergoes fission produces neutrons, which in turn split more uranium atoms. If the total mass of the uranium is sufficient, the number of neutrons increases exponentially.

A nuclear power plant: This is similar to the above example, but this time the fission process (also called *divergence of the reactor*) is controlled so that the growth, while exponential, is relatively slow.

We will introduce several examples of organisms whose growth is exponential.

Example 1. Consider the world's smallest flowering plant *Wolffia microscopica*, better known as duckweed. Since they reproduce by budding, millions of these plants can grow in a small pond. The process of budding happens quickly, and in the case of a *Wolffia microscopica* it takes place in 30 hours. According to Armstrong (2004), if one plant could keep replicating nonstop every 30 hours for four months, a nonillion (1 followed by 30 zeros) plants would result. Yet not only

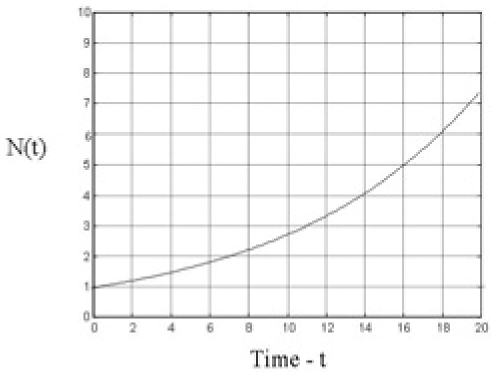

Figure 1: Exponential growth with r = 0.1 and N(0) =1

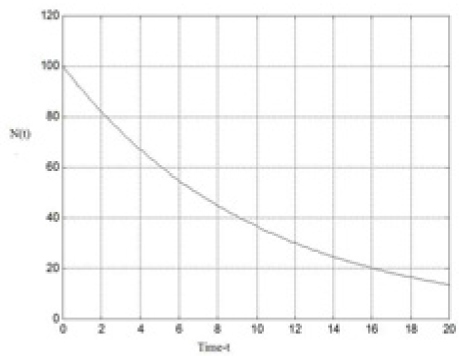

Figure 2: The graph of the exponential decay with r = -0.1 and N(0)=100

is the reproduction rate of *Wolffia microscopica* high, but also the plant's nutritional content. Duckweed contains about 40% proteins (as much as soy beans) and high levels of all essential amino acids except methionine. For these reasons, many countries, for example Thailand, harvest duckweed ponds.

From the experiment results published at "Wayne's Word, An Online Textbook of Natural History," we know that the growth rate of *Wolffia microscopica* is r =.56 plants per day (Armstrong 2004). The dynamics of *Wolffia microscopica* are described by the initial value problem for the differential equation

$$\frac{dN}{dt} = 0.5N, N(0) = N_0 \tag{2}$$

where N_0 is the number of plants at the initial moment.

The population size N(t) of *Wolffia microscopica* is a solution of the initial value problem (2) and is given by the formula

$$N(t) = N(0)e^{0.56t} \tag{3}$$

Formula (3) can be used to predict the population size on any day. For example, we can predict how many plants there will be on the 17th day if there was just one plant at the beginning. From t=17, $N_0 = 1$, and Formula (3) we obtain N(15) = exp(.56*17) = 13, 630.

Example 2. Another example of population growth is the proliferation of rabbits after their introduction to Australia. In 1859, Thomas Austin, a southern Australian farmer homesick for England, imported two dozen wild English rabbits and set them free on his land. Within six years the initial 24 rabbits had multiplied to 22 million. The dynamics of the rabbits are described by

$$\frac{dN}{dt} = 0.0063N, \text{with the initial condition } N(0) = 24$$

Results

The main results of the paper deal with exponential growth or decay with impulsive perturbations. The findings are set up as theorems and corollaries, with their applications as examples.

Let us consider a bioreactor in which instantaneous perturbations occur because an amount of the population is taken away or added. We will look at the case when the population between two consecutive perturbations is growing or decaying exponentially. This is portrayed in Figure 3.

To describe the dynamics of the population in the above-mentioned type of bioreactor we will use a new model. This model consists of three parts:

The first part describes the continuous dynamics of the process, the second part describes the instantaneous perturbations, and the last part describes the initial size of the population.

Let us assume that a population is growing or decaying exponentially with instantaneous perturbations at time $t_1 < t_2 < t_3 < ... < t_n < ...$ such that $\lim_{n\to\infty} t_n = \infty$. Then, the dynamics of the population could be described by the differential equation

$$\frac{dN}{dt} = rN \ for \ t \neq t_i, i = 1, 2, 3, ... \tag{4}$$

the impulsive condition

$$N(t_i + 0) = I_i(N(t_i - 0)) \ for \ i = 1, 2, 3, ... \tag{5}$$

and the initial condition

$$N(0) = N_0 \tag{6}$$

where $N(t)$ is the population size at the moment $t, r = constant$ is the rate of change of the population size, N_0 is the initial size of the population, $t_i, i = 1, 2, 3, ...$ are the moments at which the bioreactor is perturbed instantaneously, $N(t_i - 0)$ is the amount of the population before the perturbation at moment t_i, $N(t_i + 0)$ is the amount of the population after the perturbation at the moment t_i, and the function $I_i(N)$ gives the amount of the population size after the perturbation at the moment t_i.

The set of equations (4), (5) is called an Impulsive Differential Equation and the set of equations (4), (5), (6) is called an initial value problem for Impulsive Differential Equations (Lakshmikantham *et al.* [1]).

We note that the moments of impulses could be fixed and given before the investigation of the problem, or they could depend on some conditions and vary with the initial condition or the solution.

We will note that in the case when $I_i(N) = N$, we find using Equation (7) that $N(t_i+0) = I_i(N(t_i-0)) = N(t_i-0)$, so there is no jump at the point $t = t_i$. That is why if for all $i = 1, 2, 3, ...$ the condition $I_i(N) = N$ holds, then the Impulsive Differential Equation (4), (5) reduces to the ordinary differential equation (1), and the new model reduces to the classical model.

In the case when $I_i(N) > N$ the inequality $N(t_i + 0) > N(t_i - 0)$ holds. Therefore the population size after the perturbation is bigger than the population size before the perturbation, which means that at the moment t_i a biomass (the amount of living matter in

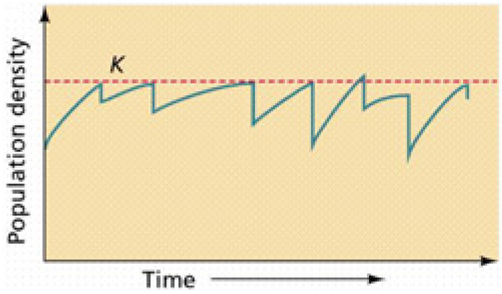

Figure 3:

a certain habitat) is added to the bioreactor. Similarly, the inequality $I_i(N) < N$ means that a biomass is taken away from the bioreactor at the moment t_i. At the same time, because the population size is a nonnegative number, the function $I_i(N)$ is nonnegative.

We will consider several particular types of perturbations to a bioreactor.

Case 1. Proportional Perturbations:

We consider the case when the impulsive changes are proportional to the existing amount before the perturbation. Then the jump condition could be written as $N(t_i+0) = B_i N(t_i-0)$, where the constant B_i is the coefficient of proportionality.

In this case, we could consider two sub-cases. *Case 1.1. Fixed moments of impulses.*

Let the impulsive moments $t_1 < t_2 < t_3 < ... < t_n <$ ...be fixed such that $\lim_{n\to\infty} t_n = \infty$. Then the impulsive model can be written in the form

$$\frac{dN}{dt} = rN \ for\ t \neq t_i, i = 1,2,3,... \quad (7)$$
$$N(t_i+0) = B_i N(t_i-0)\ for\ i = 1,2,3,... \quad (8)$$
$$N(0) = N_0 \quad (9)$$

We will note that (7), (8), (9) is a partial case of (4), (5), (6) in which $I_i(N) = B_i N$. If $0 \leq B_i < 1$then a biomass is taken away from the bioreactor at a moment t_i, and if $B_i > 1$ then a biomass is added to the bioreactor at a moment t_i. In the case $B_i = 1$ there is no jump at t_i.

The population growth for the proportional perturbations model is given in Figure 4.

First we will prove some theoretical results for the initial value problem (7), (8), (9).

Theorem 1.1.: The initial value problem (7), (8), (9) has a unique solution for $t \geq 0$, which is a piecewise continuous function with points of discontinuity at point $t_i, i = 1,2,3,...$, and which is given by the formula

$$N(t) = \left(\prod_{i=0}^{n} B_i\right) N_0 e^{rt}\ for\ t \in (t_n, t_{n+1}], n = 0,1,2,3,... \quad (10)$$

where $B_0 = 1, B_1 \geq 0, t_0 = 0, \prod_{1=0}^{n} B_i = B_0 B_1 ... B_n$

Proof. We use the method of mathematical induction to prove Formula (10).

Let $t \in (t_0, t_1]$. Then the initial value problem for the Impulsive Differential Equation (7), (8), (9) reduces to the initial value problem for the ordinary differential equation (1), (2) whose solution is $N(t) = N_0 e^{rt} = B_0 N_0 e^{rt}$. Therefore Formula (10) is true for $n = 0$.

Assume that Formula (10) is true for $n-1$ (i.e., for $t \in (t_{n-1}, t_n]$). From Equation (8) we find that

$$\begin{aligned} N(t_n+0) &= B_n N(t_n-0) \\ &= B_n \left(\prod_{i=0}^{n-1} B_i\right) N_0 e^{rt_n} \\ &= \left(\prod_{i=0}^{n} B_i\right) N_0 e^{rt_n} \end{aligned} \quad (11)$$

Let $t \in (t_n, t_{n+1}]$. Then the solution of Equation (7) is

$$\int_{t_n}^{t} \frac{dN}{N} = r\int_{t_n}^{t} dt, ln\ N(t) - ln\ N(t_n+0) = r(t-t_n) \text{ or}$$
$$N(t) = N(t_n+0)e^{r(t-t_n)}$$

Substitute (11) into the above equation and obtain

$$\begin{aligned} N(t) &= \left(\prod_{i=0}^{n} B_i\right) N_0 e^{rt_n} e^{r(t-t_n)} \\ &= \left(\prod_{i=0}^{n} B_i\right) N_0 e^{rt} \end{aligned} \quad (12)$$

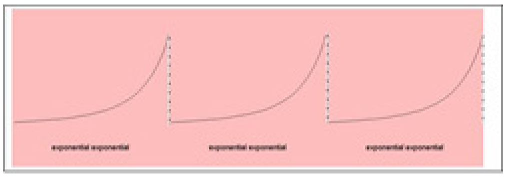

Figure 4: Exponential growth with impulsive perturbations.

Equality (12) proves the validity of (10) for $t \in (t_n, t_{n+1}]$.

Therefore Formula (10) is correct for all $n \geq 0$.

Corollary 1.1. Let $B_i = 1, i = 1, 2, 3, ...$ Then Formula (10) reduces to the solution of the classical continuous exponential model (1), (2).

Corollary 1.2. The coefficients of proportionality B_i and the population size are directly proportional.

Corollary 1.3. If $0 \leq B_i \leq 1$ for $i = 1, 2, 3, ...$ then the length of the interval of perturbations $t_{n+1} - t_n$ and the population size are directly proportional. If $B_i > 1$ for $i = 1, 2, 3, ...$ then the length of the interval of perturbations $t_{n+1} - t_n$ and the population size are indirectly proportional.

Corollary 1.4. Let $r < 0$ and $0 \leq B_i \leq 1$ for $i = 1, 2, 3, ...$ Then $\lim_{t\to\infty} N(t) = 0$ (i.e., the population size approaches 0 in time).

Proof. From Formula (10) and the inequalities $0 \leq B_i \leq 1$, we obtain that $\prod_{i=0}^{n} B_i \leq 1$ and $0 \leq N(t) \leq N_0 e^{rt}$. Taking a limit in the last inequality as t increases without bound, we find that $N(t)$ approaches 0.

Remark 1.1. According to Corollary 1.3 the solution decreases and approaches zero faster than the exponential function. That is why in practice, in order to avoid a faster decrease of the population size in the case of a negative net rate of growth, it is necessary that $B_i > 1$, which is equivalent to adding a biomass to the bioreactor.

Corollary 1.5 Let $r > 0$ and $B_i > 1$ for $i = 1, 2, 3, ...$ Then $\lim_{t\to\infty} N(t) = \infty$ (i.e., the population size increases in time without bound).

Proof. From Formula (10) and the inequalities $B_i > 1$, we obtain $\prod_{i=0}^{n} B_i > 1$ and $N(t) > N_0 e^n$. Taking a limit in the last inequality as t increases without bound, we find that $N(t)$ increases without bound.

Remark 1.2. According to Corollary 1.4, the solution increases faster than the exponential function. That is why in practice, to avoid a faster increase of the population size and an exhaustion of resources in the case of a positive net rate of growth, it is necessary that $0 \leq B_i \leq 1$, which is equivalent to taking away a biomass from the bioreactor.

Corollary 1.6. If there exists a constant $T > 0$ such that

$$t_{i+1} = t_i + T, B_i = e^{-rT}, i = 1, 2, 3, ...$$

then the initial value problem (7), (8), (9) has a periodic solution with a period T, and the perturbed amount at each impulsive moment t_i is $N_0(1 - e^{rT})$.

Proof. Let $t \in (t_n, t_{n+1}]$. Then $t + T \in (t_{n+1}, t_{n+2}]$ and $N(t+T) = \left(\prod_{i=0}^{n+1} B_i\right) N_0 e^{r(t+T)} = \left(\prod_{i=0}^{n+1} e^{-rT}\right) N_0 e^{r(t+T)} = \left(\prod_{i=0}^{n} e^{-rT}\right) N_0 e^{rt} = N(t)$.

Then the perturbed amount is $N(t_i + 0) - N(t_i - 0) = N_0(1 - e^{rT})$.

Remark 1.3. If the value of the expression is positive, then it is equivalent to adding an amount to the bioreactor. If the value is negative, then it is equivalent to removing an amount from the bioreactor.

Example 1.1. Consider an artificial pond (bioreactor) containing *Wolffia microscopica* in which there is initially one plant, and every fourth day we remove half of the existing plants from the bioreactor. Then $t_i = 4i, i = 1, 2, 3, ...$ and the dynamics of the population size of *Wolffia microscopica* are described by the initial value problem

$$\frac{dN}{dt} = 0.56N \ for \ t \neq 4i, i = 1, 2, 3, ... \quad (13)$$

$$N(t_i + 0) = 0.5N(t_i - 0) \ for \ i = 1, 2, 3, ... \quad (14)$$

$$N(0) = 1 \quad (15)$$

From Formula (10), for $B_i = .5, r = .56$, we obtain the solution

$$N(t) = \left(\prod_{i=0}^{n} 0.5\right) e^{(0.56)t} = (0.5)^i e^{(0.56)t}$$

$$for \ t \in (4i, 4i+4], i = 0, 1, 2, 3, ... \quad (16)$$

We can apply Formula (16) to predict the population size. For example, we can predict how many plants there will be on the 17th day. For this purpose we use (16) with $t = 17 \in (4i, 4i+4], i = 4$, and obtain the population size $N(17) = (0.5)^4 e^{(0.56)17}$ plants.

When compared to the classical case of exponential growth, in which N(15)=13,630, the proportional model is much more realistic. In real life population sizes are curbed instantaneously by diseases, predators, or other environmental factors. Thus, some individuals from the population are taken away, and there are jumps in the growth pattern.

We can use Formula (10) to predict the population size if we change some of the parameters.

First, we will consider changes in the time of perturbations. For example, let us assume that every 6th day half of the plants are removed. Then how many plants will there be on the 17th day?

From Formula (10), for $t_i = 6i$, t=$t = 17 \in (6i, 6i+6], i = 2$, we obtain that $N(17) = (0.5)^2 e^{(0.56)17} = 3047$. So in this case ($0 \leq B_i = 0.5 < 1$), increasing the length of the interval of plant removal involves increasing the population size.

Second, we will consider changes in the amount of perturbations. For example, let three-fourths of the plants be removed every fourth day instead of a half. Then how many plants will there be on the 17th day?

In this case the solution is

$$N(t) = \left(\prod_{i=0}^{n} 0.75\right) e^{(0.56)t} = (0.75)^i e^{(0.56)t}$$
$$for\ t \in (4i, 4i+4], i = 0, 1, 2, 3, ...$$

From the above forumula, for $t = 17 \in (4i, 4i+4], i = 4$ we obtain $N(17) = (0.75)^4 e^{0.56)17} = 4312$. So increasing the coefficient of proportionality involves increasing the population size.

If we are interested in having a periodic process with a period of four days, then from Corollary 4.1.6, $B_i = e^{(-0.56)4} = e^{-2.24}$, and we have to remove an amount $e^{2.24} - 1 \approx 8$ plants every fourth day.

Case 1.2. Variable moments of impulses.

Let us consider the case when the impulsive perturbations occur at moments at which the population size satisfies a given condition. For example, they might occur when the population size is twice the initial amount, or when it is equal to a given number. In this case, the impulsive moments t_n are not given from the beginning, and they depend on the solution of the initial value problem. For example, let the perturbations in the bioreactor occur when the population size reaches a given level L. Then the dynamics are described by the initial value problem (7), (8), (9) where the points of jumps t_n are solutions of the equations $N(t_n - 0) = L$.

Theorem 1.2. Let one of the following conditions be satisfied:

$r > 0$ and $0 < B_n < 1, \lim_{n\to\infty} \prod_{j=0}^{n} B_j = 0$
(ii) $r < 0$ and $B_n > 1, \lim_{n\to\infty} \prod_{j=0}^{n} B_j = \infty$

Then the initial value problem (17), (18), (19) with impulses at $t_n : N(t_n) = L$ has a unique solution for $t \geq 0$, given by Formula (10), where the impulsive moments are

$$t_n = \frac{1}{r} ln \frac{L}{N_0 \prod_{j=0}^{n-1} B_j}, n = 1, 2, 3, ... \qquad (17)$$

and the amount of jump at each point t_n is $|B_n - 1|\, L$. Additionally, $lim_{n\to\infty} t_n = \infty$.

The proof is based on the application of mathematical induction, and it is similar to the proof of Theorem 1.1.

Corollary 1.7. In the case of constant proportional perturbations, such that $B_n = B, n = 1, 2, ...$, the conditions of Theorem 1.2 are reduced to the following conditions:

(i) $r > 0$ and $0 < B < 1$;
(ii) $r < 0$ and $B > 1$.

Corollary 1.8. If $r > 0$ and any increases then the length of the time interval until the next perturbation decreases. Conversely, if $r < 0$ and any B_i increases, then the length of the time interval until the next perturbation increases.

Example 1.2. Consider an artificial pond (bioreactor) containing *Wolffia microscopica*, in which there is initially one plant, and in which the population size must always be less than 1,000 plants. One of the most important questions in the eco-control is when is the best time to remove plants from the bioreactor, and how many?

From Theorem 1.2 we find that the first moment of perturbation is

$t_1 = \frac{1}{r} ln \frac{L}{N_0} = \frac{1}{0.56} ln 1000 \approx 12\ days$, and the maximum amount taken away from the bioreactor is 999 plants, so $999 = |B - 1|\, 1000$, or B=0.001. Then the next impulsive moments are

$t_2 = \frac{1}{0.56} ln \frac{1000}{0.001} \approx 25\ days$, $t_3 = \frac{1}{0.56} ln \frac{1000^2}{999^2}$, $t_4 \approx 49\ days$. Theoretically, this method allows us to find how to control the system in order to keep the population size less than 1,000 plants.

Case 2. Non-proportional Perturbations.

Now we consider the case when the impulsive changes do not depend on the existing population size before the perturbation (i.e., the impulsive changes are constants). Then the jump condition could be written as $N(t_i+0) = B_i + N(t_i-0)$, where the constants B_i are the amount of perturbation to the existing population size.

We will consider two cases:

Case 2.1. Fixed moments of impulses.

Let the impulsive moments $t_1 < t_2 < t_3 < ... < t_n < ...$ be fixed such that $\lim_{n\to\infty} t_n = \infty$. Then the impulsive model can be written in the form

$$\frac{dN}{dt} = rN \ for \ t \neq t_i, i = 1, 2, 3, ... \quad (18)$$

$$N(t_i + 0) = B_i + N(t_i - 0) \ for \ i = 1, 2, 3... \quad (19)$$

$$N(0) = N_0 \quad (20)$$

Where B_i are constants giving the amount of perturbation at moment t_i.

We will note that (18), (19), (20) is a partial case of (4), (5), (6) in which $I_i(N) = B_i + N$. It is necessary to have $B_i \geq -N(t_i - 0), i = 1, 2, 3, ...$ since the population size is always positive. So, if $-N(t_i - 0) \leq B_i < 0$, then a biomass is taken away from the bioreactor at the moment t_i, and if $B_i > 0$ then a biomass is added to the bioreactor at the moment t_i. In the case $B_i = 0$, there is no jump at t_i.

Theorem 2.1. The initial value problem (18), (19), (20) has a unique solution for $t \geq 0$ which is a piecewise continuous function with points of discontinuity at $t_i, i = 1, 2, 3, ...$ and which is given by the formula

$$N(t) = \sum_{i=0}^{n} B_i e^{r(t-t_i)} \ for \ t \in (t_n, t_{n+1}], n = 0, 1, 2, 3, ... \quad (21)$$

where $B_0 = N_0, t_0 = 0, \sum_{i=0}^{n} B_i = B_0 + B_1 + ... + B_n, B_n \geq -\sum_{i=0}^{n-1} B_i e^{r(t-t_i)}$.

Proof. The base of the proof is using mathematical induction and the formula $N(t) = N(t_n + 0)e^{r(t-t_n)} = (B_n + N(t_n - 0))e^{r(t-t_n)}, t \in (t_n, t_{n+1}]$.

Corollary 2.1. If $B_i = 0, i = 1, 2, 3, ...$ then Formula (21) reduces to the solution of the classical continuous exponential model (1), (2).

Corollary 2.2 If $r > 0$ and $B_i > 0, i = 1, 2, 3, ...$ then $\lim_{t\to\infty} N(t) = \infty$. If $r < 0$, and $B_i < 0, i = 1, 2, 3, ...$ then $\lim_{t\to\infty} N(t) = 0$.

Remark 2.1. According to Corollary 2.2, if the net rate of growth is positive, then the solution increases faster than the exponential function and approaches infinity in time. To avoid this increase, which is connected with the exhaustion of resources, it is necessary to keep $B_i < 0, i = 1, 2, 3, ...$ This is equivalent to taking away a biomass at the impulsive moments.

At the same time, if the net rate of exponential growth is negative, then to keep the population size from approaching 0 we must keep $B_i > 0, i = 1, 2, 3, ...$, which is equivalent to adding a biomass to the bioreactor at the impulsive moments.

Corollary 2.3. The amount of perturbations B_i and the population size are directly proportional.

Corollary 2.4. Let there exist a number T such that $t_{n+1} = t_n + T, n = 0, 1, 2, ...$ and $B_n = N_0(1 - e^{rT}), n = 1, 2, 3, ...$

Then the initial value problem (18), (19), (20) has a periodic solution with a period T.

Proof. Let $t \in (T_n, t_{n+1}]$. Then $t + T \in (t_{n+1}, t_{n+2}]$ and $e^{r(t+T-t_i} = e^{r(t-t_{i-1})}, i = 1, 2, 3, ...$

From Formula (21) we obtain

$$N(t+T) - N(t) = \sum_{i=0}^{n+1} B_i e^{r(t+T-t_i)} - \sum_{i=0}^{n} B_i e^{r(t-t_i)} = N_0 e^{r(t+T)} +$$

which proves the periodicity of the solution.

Example 2.1. Consider an artificial pond (bioreactor) containing *Wolffia microscopica*, in which there is initially one plant, and from which we remove five plants every fourth day. Then points of instantaneous perturbations are 4, 8, 12,..., i.e., $t_i = 4i, i = 1, 2, 3, ...$ and the constant amount $B_i = 5$. In this case the dynamics of the population size of *Wolffia microscopica* are described by the initial value problem

$$\frac{dN}{dt} = 0.56N \ for \ f \neq t_i, i = 1, 2, 3, ... \quad (22)$$

$$N(t_i + 0) = N(t_i - 0) - 5 \ for \ i = 1, 2, 3, ... \quad (23)$$

$$N(0) = 1 \quad (24)$$

From Theorem 2.1 and Formula (21), with $B_i = -5, r = 0.56$ we obtain the solution

$$N(t) = e^{0.56t} - \sum_{i=0}^{n} 5e^{0.56(t-4i)} \ for \ t \in (4n, 4n+4], n = 0, 1, 2, 3, ... \quad (25)$$

We will now apply Formula (25) to predict the population size. For example, how many plants will there be on the 17th day?

From $17 \in (4i, 4i+4], i = 4$ and Formula (25) we find that

$$N(17) = e^{(0.56)17} - 5(e^{0.56(17-4)} + e^{0.56(17-8)} + e^{0.56(17-12)} + e^{0.56(1}$$

We can find the maximum constant amount which could be removed from the bioreactor every fourth day using the obtained theoretical result.

From the inequality $B_i \leq -N(t_i - 0)$, where $B_i = B$, and $N(t_1 - 0) = e^{(0.56)4}$ we can conclude that nine is the maximum number of plants that could be taken away.

Let us change the perturbations' length. For example, consider a period of six days. In this case, $t_1 = 6, N(t_1 - 0) = e^{(0.56)6} = 28.8$ and we find that the maximum amount that could be taken away is 28 plants. Note that if we increase the length of the time interval of perturbation by two days, it results in an additional $28 - 9 = 19$ plants.

If we need, for example, to ensure that a periodic process has an interval of five days, then T=5, $B \approx -15$, which means every fifth day 15 plants will be removed.

0.0.1 Case 2.2. Variable moments of impulses.

Let us consider the case when the impulsive perturbations occur when the population size satisfies a given condition. For example, let the perturbations in the bioreactor occur when the population size reaches a given level L. Then the dynamics are described by the initial value problem (18), (19), (20), where the points of jumps t_i are solutions of the equations $N(t_i - 0) = L$.

Theorem 2.2. Let one of the following conditions be satisfied:

(i) $r > 0$ and $\lim_{n\to\infty} \prod_{j=1}^{n} \left(1 + \frac{B_j}{L}\right) = 0, 0 > B_n \geq -L$, for $n = 0, 1, 2, ...$
(ii)$r < 0$ and $\lim_{n\to\infty} \prod_{j=1}^{n} \left(1 + \frac{B_j}{L}\right) = \infty, 0 < B_n < \infty$, for $n = 0, 1, 2, ...$

Then the initial value problem (18), (19), (20) with impulses at $t_n : N(t_n - 0) = L$ has a unique solution for $t \geq 0$, given by

$$N(t) = N_0 e^{rt} \prod_{i=0}^{n} \left(1 + \frac{B_i}{L}\right), t \in (t_n, t_{n+1}], n = 0, 1, 2, ... \tag{26}$$

where the moments of impulses are

$$t_n = \frac{1}{r} ln \frac{L^n}{N_0 \prod_{j=1}^{n-1}(B_j + L)}, n = 1, 2, 3, ...$$
$$\text{and } lim_{n\to\infty} t_n = \infty \tag{27}$$

Proof. The proof is similar to the proof of Theorem 1.1 and we will omit it.

In the case of the same perturbations at each moment, the conditions in Theorem 2.2 can be written in a simpler way.

Corollary 2.4. Let $B_n = B, n = 1, 2, 3, ...$ and

(i) $r > 0$ and $-L \leq B < 0$
(ii)$r < 0$ and$0 < B < \infty$.

Then $\lim_{n\to\infty} t_n = \infty$, where the moments of impulses are$t_n = \frac{1}{r} ln \frac{L}{N_0} \left(\frac{L}{B+L}\right)^{n-1}$.

Remark 2.2. From Theorem 2.2 it follows that in the case of a positive rate of growth. The amount of perturbation is negative, which is equivalent to removal of a biomass. In the case of a positive rate of growth, the amount of perturbation is positive, which is equivalent to adding a biomass.

Corollary 2.5. If $r > 0$ and we increase any B_i, then the length of the time interval until the next perturbation decreases. Conversely, if $r < 0$ and we increase any B_i, then the length of the time interval until the next perturbation increases.

Example 2.2. Consider an artificial pond (bioreactor) containing *Wolffia microscopica*, in which there is initially one plant, and in which the population size must be less than 1,000 plants. We can use the obtained theoretical results to find the best moments for removal of plants from the bioreactor, and the proper amount to be removed.

From Formula (27) we find that the first moment of perturbation
is $t_1 = \frac{1}{r} ln \frac{L}{N_0} = \frac{1}{0.56} ln 1000 \approx 12^{th}\ day$.

If we need to remove 900 plants, then the next moments of removal, according to Formula (29), are

$$t_2 = \frac{1}{0.56} ln \frac{1000^2}{1000 - 900} \approx 16^{th}\ day$$
$$t_3 = \frac{1}{0.56} ln \frac{1000^3}{(1000 - 900)^2} \approx 21^{st}\ day$$

If we need to remove 999 plants at each impulsive moment, then

$$t_2 = \frac{1}{0.56} ln \frac{1000^2}{1000 - 900} \approx 25^{th}\ day$$
$$t_3 = \frac{1}{0.56} ln \frac{1000^3}{(1000 - 900)^2} \approx 37^{th}\ day$$

Therefore, if we need more plants to be removed, then we must take these plants away less frequently.

Discussion and Conclusions

In the description of exponential growth or decay of impulsive systems, the behavior of a system depends on the type of the perturbation. Since there are many kinds of instantaneous changes, we restricted our investigation to two cases: proportional and constant perturbations.

In the first case, the perturbed amount is proportional to the existing population size. In the second case, each perturbation amount is independent from the existing population size. In both instances, two sub-cases are investigated: when the impulsive moments are fixed from the beginning, and when the perturbations occur at the instants at which the population size reaches a certain level. Theorem 1.2 and 2.2 allow us to control the system so that the population size is never above or below a given level. Formulas (10) and (20) for the solutions of the models can be used to predict the population size in time, the instant perturbations occur, and the number of perturbations. Corollary 1.6 and 2.4 provide a means to keep the population dynamics periodic. Corollaries 1.4, 1.5, and 2.2 give us the moments and amounts at which to influence the systems in order to avoid a fast decrease of the population size and the approach of extinction, or to avoid a fast increase of the population size and an exhaustion of resources. Note that these models can be used in the case of instantaneous changes with negligible length.

The obtained theoretical results for the new models potentially let us control the ecosystem, saving time and money for making experiments and measuring the population size. The models will be beneficial to many aspects of life. For example, artificial growth of plants, controlled breeding of fish and livestock with population size perturbed by diseases and human interference, and disease development in humans, where the growth of infected cells is perturbed by the repeated injection of medicine into the bloodstream.

Thus, without doing the experiment, we can theoretically decide the best moments of perturbations to the bioreactor and the best amounts of perturbations.

The examples given in this paper illustrate the possible applications of the obtained theoretical results. The findings can be used not only for work with Wolffia microscopica, but for any population whose net rate of growth is known.

In doing such work, we never imagined that a simple ecological model may lead to such interesting and complex outcomes. Population systems can be very intricate, but mathematical biology helps outline the growth patterns through the language of theorems. Even though the dynamics of real populations are very complicated and depend on many environmental factors, ecologists often find mathematical models useful for contemplating the extremes of the wide range of possibilities for millions of different populations.

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